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THE SOLUTION OF A SPECIAL SET OF HERMITIAN TOEPLITZ LINEAR EQUA--ETC(U)
JAN 75 D C FARDEN N00014-67-A-0299-0019

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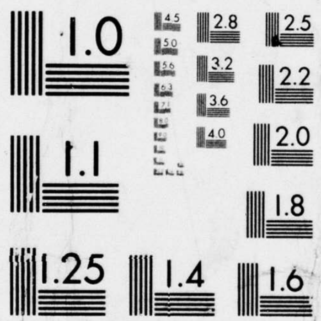

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The Solution of a Special
Set of Hermitian Toeplitz Linear Equations

by

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ABSTRACT

The solution of a set of m linear equations $L_m s_m = d_m$, where L_m is an mth order Hermitian Toeplitz matrix and the elements of d_m possess a Hermitian symmetry, is considered. A specialized algorithm is developed for this case which solves for s_m in approximately $1.5m^2$ "operations," whereas the Hermitian case of an algorithm developed by Zohar solves for s_m in approximately $2m^2$ "operations." An "operation" is used here to denote one addition and one multiplication. A further reduction in computational requirements is shown in case L_m and d_m are real. As with Zohar's algorithm, the specialized algorithm requires that all principal minors of L_m be nonzero.

KEY WORDS AND PHRASES: Linear algebra, linear equations, Toeplitz matrix, computer programming

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1. Introduction

Consider the set of linear equations

$$L_m s_m = d_m. \tag{1}$$

Zohar [1] makes use of the Trench algorithm [2], [3] to develop an efficient algorithm for solving (1) when s_m, d_m are mx1 matrices and L_m is a non-Hermitian mth-order Toeplitz matrix. In this paper, an efficient algorithm is developed for solving (1) when L_m is a Hermitian Toeplitz matrix and d_m satisfies

$$d_m^* = \hat{d}_m , \tag{2}$$

where the symbol ^ is used to denote the reversed ordering of the elements of d_m, i.e., $(\hat{d}_m)_{i,1} = (d_m)_{m+1-i,1}$ and * denotes complex conjugate. Such a specialized case can arise, for example, in the design of digital filters, as discussed in [4]. The following example serves to illustrate how such a system of equations can arise.

EXAMPLE. Let $\alpha(t)$, $\beta(t)$, $\gamma(t)$ be jointly wide-sense stationary complex-valued stochastic processes with $\alpha(t) = \beta(t) + \gamma(t)$, where $E\{\beta(t)\gamma^*(s)\} = 0$ for all real t and s and $E\{\cdot\}$ denotes statistical expectation. On the basis of the observation vector $a_m(k)$, $\tilde{a}_m(k) = [\alpha(k) \quad \alpha(k-1) \cdots \alpha(k-m+1)]$, where the symbol ~ denotes matrix transpose, it is desired to compute a linear minimum mean-square error (MMSE) estimate of $\beta(k-p)$, i.e., it is desired to minimize the quantity $E\{|\tilde{s}_m a_m(k) - \beta(k-p)|^2\}$ with respect to s_m. It is easily shown that the desired solution, s_m, satisfies (1), with $L_m = E\{a_m^*(k)\tilde{a}_m(k)\}$ and $d_m = E\{\beta(k-p)a_m^*(k)\}$. Since $\alpha(t)$ is wide-sense stationary, L_m is a Hermitian Toeplitz matrix. With $p = (m+1)/2$, it is easily seen that (2) is satisfied since $\beta(t)$ and $\gamma(t)$ are jointly wide-sense stationary..

A useful consequence of the assumptions that $d_m^* = \hat{d}_m$ and $L_m^* = \tilde{L}_m$ is that $s_m^* = \hat{s}_m$. Define E_m to be the mxm exchange matrix of Zohar [3], i.e., $E_m a_m = \hat{a}_m$ for any mx1 matrix a_m. Note that $E_m E_m = I_m$, where I_m is the mxm identity matrix. Since L_m is persymmetric [3], $E_m L_m E_m = \tilde{L}_m$. Since $\hat{d}_m = d_m^*$, from (1) we have $E_m L_m (E_m E_m) s_m = L_m^* s_m^*$, so that $\tilde{L}_m \hat{s}_m = L_m^* s_m^*$, i.e., $\hat{s}_m = s_m^*$.

The specialized algorithm developed in Section 3 of this paper solves (1) with d_m satisfying (2) in approximately $1.5m^2$ complex "operations," whereas the Hermitian case of Zohar's algorithm [1] uses approximately $2m^2$ complex "operations." An "operation" is used here to denote one addition and one multiplication. In case L_m,d_m (and hence s_m) are real, the results of Section 3 can be used to solve (1) in approximately $1.25m^2$ real multiplications and $1.5m^2$ real additions.

Both Zohar's algorithm [1] and the specialized algorithm developed in Section 3 make use of Phase 1 of the Trench algorithm [1]-[3]. Rather than review the results necessary for the development of Section 3, it is assumed that the reader is familiar with the work of Zohar [3].

2. Preliminaries

Since the techniques used in this paper are inherently related to those used by Zohar [1], an attempt is made to follow the same notational conventions. Greek letters are used for scalars, capital letters for square matrices, and lower-case letters for column matrices. Subscripts used on matrices are used to denote the number of elements in one column of the matrix.

Since Phase 1 of the Trench algorithm requires that all principal minors of L_m be nonzero, it is assumed that (1) has been normalized so that L_m has ones along its main diagonal.

3. The Specialized Algorithm

Consider the system of equations $L_m s_m = d_m$, where L_m is an mth order normalized Hermitian Toeplitz matrix and $d_m^* = \hat{d}_m$, so that d_m may be written as $\tilde{d}_m = [\xi_{\frac{m+1}{2}} \cdots \xi_2\ \xi_1\ \xi_2^* \cdots \xi_{\frac{m+1}{2}}^*]$ for m odd and

$\tilde{d}_m = [\ \xi_{\frac{m}{2}} \cdots \xi_2\ \xi_1\ \xi_1^* \cdots \xi_{\frac{m}{2}}^*]$ for m even. For m even or odd we may write

$\tilde{d}_{i+2} = [\xi_{[\frac{i+3}{2}]}\ \tilde{d}_i\ \xi^*_{[\frac{i+3}{2}]}]$, for $i = 1, 2, \cdots, m-2$ where $[x]$ denotes the largest integer less than or equal to x. The Hermitian Toeplitz nature of L_m enables us to write

$$L_{i+2} = \begin{bmatrix} 1 & \tilde{r}_{i+1} \\ r^*_{i+1} & L_{i+1} \end{bmatrix} = \begin{bmatrix} L_{i+1} & \hat{r}_{i+1} \\ \tilde{\hat{r}}^*_{i+1} & 1 \end{bmatrix}, \qquad (3)$$

where $\tilde{r}_{i+1} = [\rho_1 \rho_2 \cdots \rho_{i+1}]$ $(0 \le i \le m-2)$. Clearly, (3) may be rewritten as

$$L_{i+2} = \begin{bmatrix} 1 & \tilde{r}_i & \\ r_i^* & L_i & \hat{r}_{i+1} \\ & \tilde{\hat{r}}^*_{i+1} & 1 \end{bmatrix}.$$

Defining $L_{i+2}\, s_{i+2} = d_{i+2}$ $(1 \le i \le m-2)$, we have $L_{i+2}\left\{ s_{i+2} - \begin{bmatrix} 0 \\ s_i \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} \theta_i \\ 0_i \\ \theta_i^* \end{bmatrix}$,

where $\theta_i = \xi_{[\frac{i+3}{2}]} - \tilde{r}_i s_i$ and 0_i is an ix1 column matrix of zeros.

Defining $B_{i+2}=L_{i+2}^{-1}$, we obtain

$$s_{i+2} = \begin{bmatrix} 0 \\ s_i \\ 0 \end{bmatrix} + B_{i+2} \begin{bmatrix} \theta_i \\ 0_i \\ \theta_i^* \end{bmatrix} \qquad (4)$$

Since the inverse of a Hermitian persymmetric matrix is a Hermitian persymmetric matrix [3], B_{i+2} may be expressed in the form

$$B_{i+2} = \lambda_{i+1}^{-1} \begin{bmatrix} 1 & \tilde{e}_{i+1} \\ e_{i+1}^* & M_{i+1} \end{bmatrix} = \lambda_{i+1}^{-1} \begin{bmatrix} P_{i+1} & \hat{e}_{i+1} \\ \hat{\tilde{e}}_{i+1}^* & 1 \end{bmatrix} .$$

Letting $f_i = [I_i \; 0_i] \, e_{i+1}$, we may write

$$B_{i+2} = \lambda_{i+1}^{-1} \begin{bmatrix} 1 & \tilde{f}_i & \\ f_i^* & Q_i & \hat{e}_{i+1} \\ & \hat{\tilde{e}}_{i+1}^* & 1 \end{bmatrix} . \qquad (5)$$

Substituting (5) into (4) we obtain the result

$$s_{i+2} = \begin{bmatrix} 0 \\ s_i \\ 0 \end{bmatrix} + \lambda_{i+1}^{-1}\theta_i \left\{ \begin{bmatrix} 1 \\ e_{i+1}^* \end{bmatrix} + \theta_i^* \theta_i^{-1} \begin{bmatrix} \hat{e}_{i+1} \\ 1 \end{bmatrix} \right\} . \qquad (6)$$

In order to make use of this result, we apply the recursive relationships for Phase 1 of the Trench algorithm [1]:

Initial values: $e_1 = -\rho_1$, $\lambda_1 = 1 - |\rho_1|^2$

Recursive relationships: $\eta_i = -\rho_{i+1} - \tilde{e}_i \hat{r}_i$,

$$e_{i+1} = \begin{bmatrix} e_i + \eta_i \lambda_i^{-1} \hat{e}_i^{*} \\ \\ \eta_i \lambda_i^{-1} \end{bmatrix}, \quad \lambda_{i+1} = \lambda_i - |\eta_i|^2 \lambda_i^{-1} .$$

Finally, Phase 1 of the Trench algorithm and (6) may be combined by noting that

$$s_1 = \xi_1 \tag{7}$$

and

$$s_2 = (1 - |\rho_1|^2)^{-1} \begin{bmatrix} \xi_1 - \rho_1 \xi_1^{*} \\ \\ \xi_1^{*} - \rho_1^{*} \xi_1 \end{bmatrix} . \tag{8}$$

An immediate consequence of (6), (7), and (8) is that $s_{i+2}^{*} = \hat{s}_{i+2}$ since λ_{i+1} is real-valued. Consequently, there are two sources of increased computational speed in the specialized algorithm: (i) s_{i+2} need only be computed for $i=1,3,5,\cdots,m-2$ when m is odd and for $i=2,4,6,\cdots,m-2$ when m is even, and (ii) approximately half ($[\frac{i+3}{2}]$) of the elements of s_{i+2} need to be computed using (6), the remaining elements being obtained from the relationship $s_{i+2}^{*} = \hat{s}_{i+2}$. The following is a summary of the algorithm.

PROBLEM FORMULATION: $L_m s_m = d_m, L_m = \begin{bmatrix} 1 & \tilde{r}_{m-1} \\ \\ r_{m-1}^{*} & L_{m-1} \end{bmatrix}$,

$\tilde{r}_i = [\rho_1 \rho_2 \cdots \rho_i]$ $(1 \leq i \leq m-1)$,

$\tilde{d}_{i+2} = [\ \xi_{[\frac{i+3}{2}]} \ \tilde{d}_i \ \xi^{*}_{[\frac{i+3}{2}]}\ ]$, $s_m = ?$

Initial values: $e_1 = -\rho_1$, $\lambda_1 = 1-|\rho_1|^2$,

$$s_1 = \xi_1 \ , s_2 = \lambda_1^{-1} \begin{bmatrix} \xi_1 - \rho_1 \ \xi_1^* \\ \\ \xi_1^* - \rho_1^* \ \xi_1 \end{bmatrix}$$

Recursive relations: Compute η_i, e_{i+1}, and λ_{i+1} for $i=1,2,\cdots, m-2$. Compute θ_i and s_{i+2} for $i = 1,3,5,\cdots, m-2$ for m odd and $i = 2,4,6,\cdots, m-2$ for m even.

$$\eta_i = -\rho_{i+1} - \tilde{e}_i \hat{r}_i$$

$$e_{i+1} = \begin{bmatrix} e_i + \eta_i \lambda_i^{-1} \hat{e}_i^* \\ \\ \eta_i \lambda_i^{-1} \end{bmatrix}$$

$$\lambda_{i+1} = \lambda_i - |\eta_i|^2 \lambda_i^{-1}$$

$$\theta_i = \xi_{[\frac{i+3}{2}]} - \tilde{r}_i s_i$$

$$s_{i+2} = \begin{bmatrix} 0 \\ s_i \\ 0 \end{bmatrix} + \lambda_{i+1}^{-1} \theta_i \left\{ \begin{bmatrix} 1 \\ e_{i+1}^* \end{bmatrix} + \theta_i^* \theta_i^{-1} \begin{bmatrix} \hat{e}_{i+1} \\ 1 \end{bmatrix} \right\}$$

Making use of the fact that only $[\frac{i+3}{2}]$ elements of s_{i+2} need be computed, the above algorithm requires approximately $1.5m^2$ additions and $1.5m^2$ multiplications for the solution of s_m. This compares with $2m^2$ for the Hermitian case of Zohar's algorithm [1].

In case L_m, d_m (and hence s_m) are real, an even further reduction in computational requirements results. For this case (6) may be rewritten as

$$s_{i+2} = \begin{bmatrix} 0 \\ s_i \\ 0 \end{bmatrix} + \lambda_{i+1}^{-1} \ \theta_i \left\{ \begin{bmatrix} 1 \\ \\ e_{i+1} \end{bmatrix} + \begin{bmatrix} \hat{e}_{i+1} \\ \\ 1 \end{bmatrix} \right\} , \qquad (9)$$

and the computation of $\tilde{r}_i s_i$ in the expression for θ_i may be computed as

$$\tilde{r}_i s_i = \sum_{\ell=1}^{i/2} (s_i)_\ell (\rho_\ell + \rho_{i+1-\ell}) \tag{10}$$

for i even and

$$\tilde{r}_i s_i = \sum_{\ell=1}^{\frac{i-1}{2}} (s_i)_\ell (\rho_\ell + \rho_{i+1-\ell}) + (s_i)_{\frac{i+1}{2}} \rho_{\frac{i+1}{2}} \tag{11}$$

for i odd. Making use of these expressions, the specialized algorithm requires approximately $1.5m^2$ additions and $1.25m^2$ multiplications. A slightly different form of (9) can be easily obtained as

$$s_{i+2} = \begin{bmatrix} 0 \\ s_i \\ 0 \end{bmatrix} + \frac{\theta_i}{\lambda_i - \eta_i} \begin{bmatrix} 1 \\ e_i + \hat{e}_i \\ 1 \end{bmatrix} . \tag{12}$$

This final expression (12) is slightly more efficient than (9). A FORTRAN routine for the specialized algorithm making use of (10)-(12) is presented in [5].

EXAMPLE. Let $\rho_i = (i+1)^{-1}$ for $i=1,2,\cdots, m-1$ and $\xi_i = i^{-1}$, for $i=1,2,\cdots, [\frac{m+1}{2}]$. A FORTRAN routine, called TPSLV, based on the symmetric case of [1] was written for a timing comparison with the FORTRAN routine, called SYMM, presented in [5]. The time needed (in seconds) for each routine to compute s_m for this example with $m \in \{10,50,100,500\}$ is indicated in the following table.

M	TPSLV	SYMM
10	.005	.005
50	.089	.057
100	.343	.217
500	8.266	5.233

The above results, obtained on a CDC 6400 computer, agree with the computational considerations presented above.

4. Concluding Remarks

An algorithm has been developed for the solution of a specialized set of Toeplitz linear equations that arise in linear filtering applications. The savings in computational requirements of the new algorithm over the results of Zohar [1] are approximately 25% for the Hermitian case and 37.5% for the real case. Finally, it is noted that the techniques used in developing the specialized algorithm can indeed be applied to the general case treated by Zohar [1]; however, such a development results in an algorithm having no computational advantage over the generalized algorithm of [1].

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The solution of a set of m linear equations $L_m s_m = d_m$, where L_m is an mth order Hermitian Toeplitz matrix and the elements of d_m possess a Hermitian symmetry, is considered. A specialized algorithm is developed for this case which solves for s_m in ~~approximately~~ about $1.5m^2$ operations, whereas the Hermitian case of an algorithm developed by Zohar solves for s_m in approximately $2m^2$ → next page

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